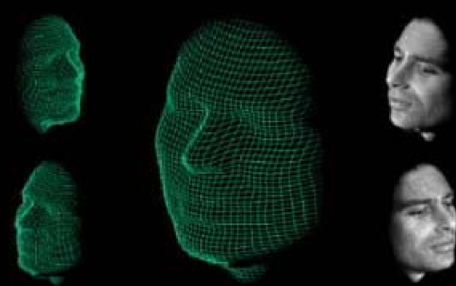
Computer Science
Department

Technion-Israel Institute of Technology

Numerical Geometry

Ron Kimmel



Information Society
Technologies

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European FP 6 NoE
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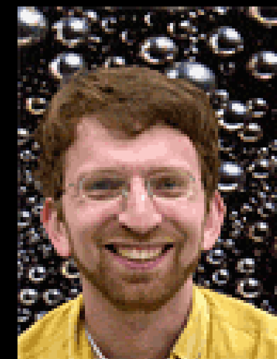
Students

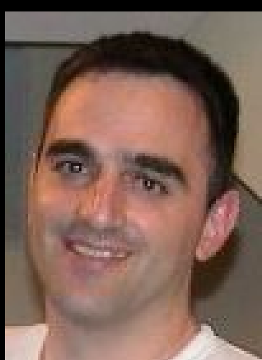

Michal Aharon
CS Technion


Ilya Blayvas
CS Technion


Alexander Bronstein
EE-CS Technion

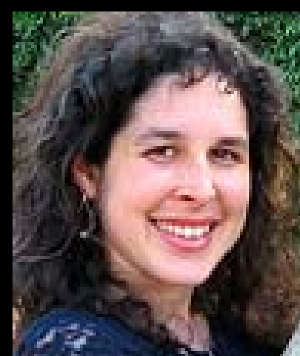

Michael Bronstein
CS Technion


Alexander Brook
Math Technion


Asi Elad
IO-Image


Roman Goldenberg
CS Technion


Eyal Gordon
CS Technion


Michal Holzman Gazit
EE-CS Technion


Alon Spira
CS Technion


Gil Zigelman
3DV Systems

Colleagues

Alfred M. Bruckstein
CS Technion

Laurent Cohen
Paris Dauphine

Vicent Caselles
Math. Barcelona

Nahum Kiryati
Tel Aviv Univ.

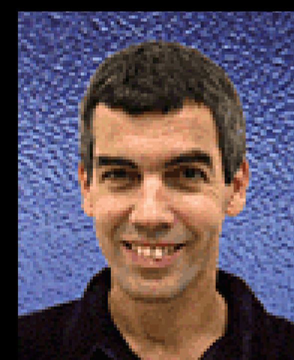
Ehud Rivlin
CS Technion

Michael Rudszky
CS Technion

Guillermo Sapiro
EE Minnesota

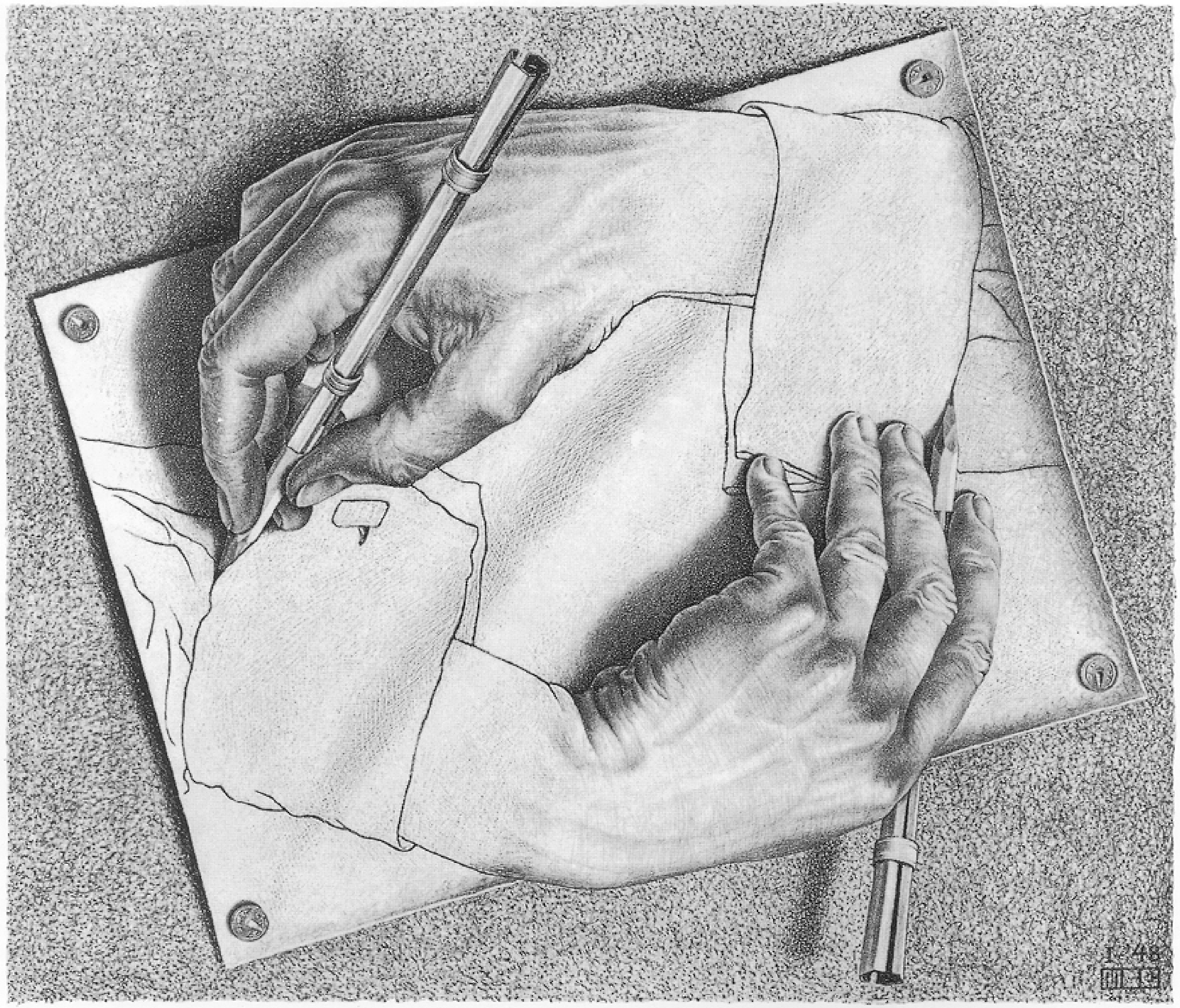
James Sethian
UC Berkeley

Nir Sochen
Tel Aviv Univ.



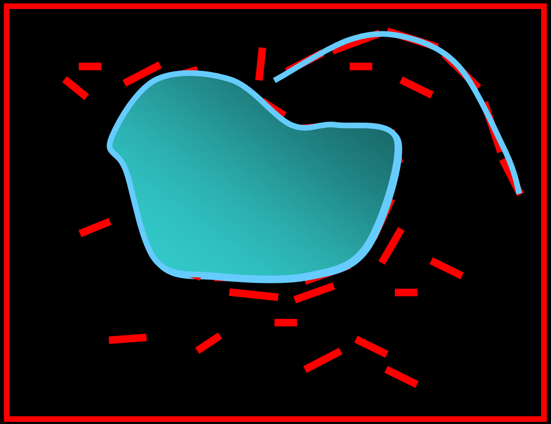
Irad Yavneh
CS Technion

On Edge Detection and Integration



Edge Detection

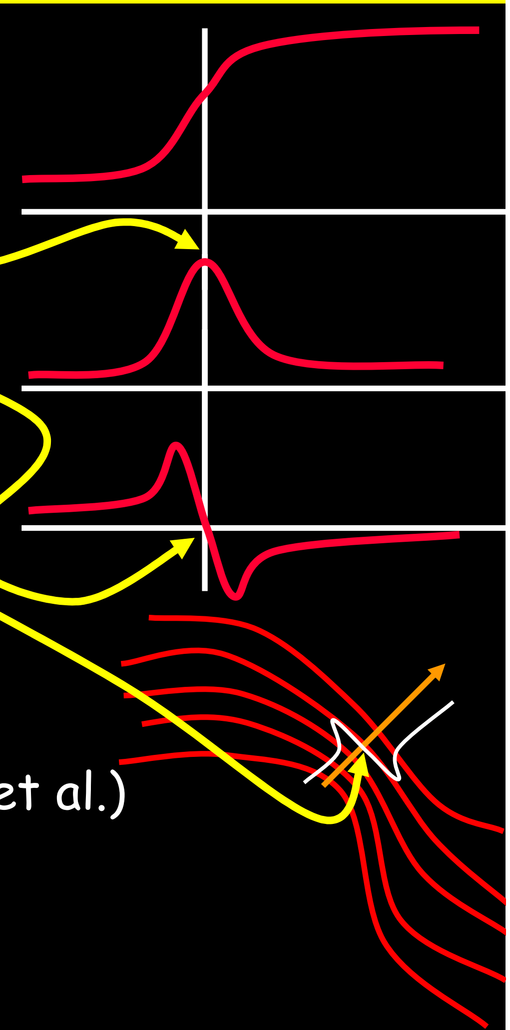
- Edge Detection:
  - The process of labeling the locations in the image where the gray level's "rate of change" is high.
    - OUTPUT: "edgels" locations, direction, strength



- Edge Integration:
  - The process of combining "local" and perhaps sparse and non-contiguous "edgel"-data into meaningful, long edge curves (or closed contours) for segmentation
    - OUTPUT: edges/curves consistent with the local data

The Classics

- Edge detection:
  - Sobel, Prewitt, Other gradient estimators
  - Marr Hildreth
    zero crossings of $\Delta G * I$
  - Haralick/Canny/Deriche et al.
    "optimal" directional local max of derivative
    $I_{\xi\xi} = 0$

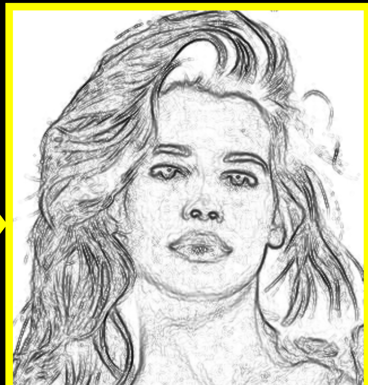


- Edge Integration:
  - tensor voting (Rom, Medioni, Williams, ...)
  - dynamic programming (Shashua & Ullman)
  - generalized "grouping" processes (Lindenbaum et al.)

The "New-Wave"

- Snakes
- Geodesic Active Contours
- (Variational) Model Driven Edge Detection

Image



Edge Indicator Function

$$g(x,y) = \frac{1}{1+|\nabla(G_\sigma * I)|^2}$$

"nice" curves that optimize a functional of g( ), i.e.

$$\int_{curve} g(\,)ds$$

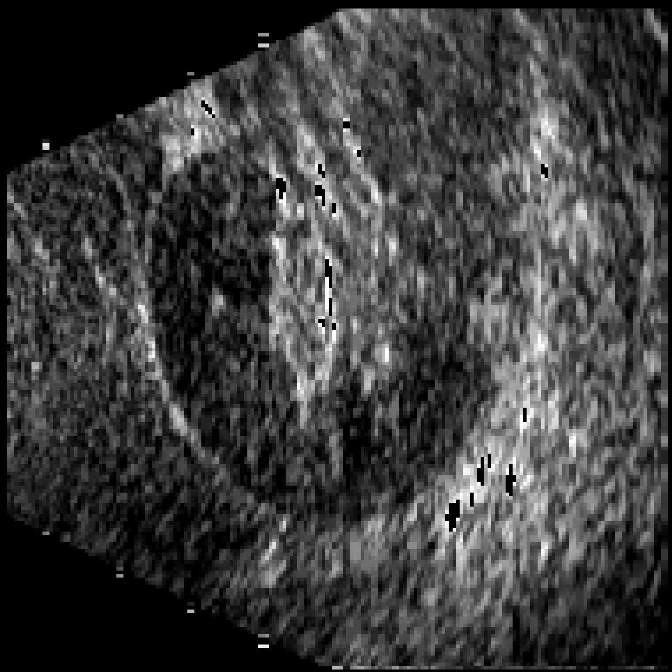
nice: "regularized", smooth, fit some prior information

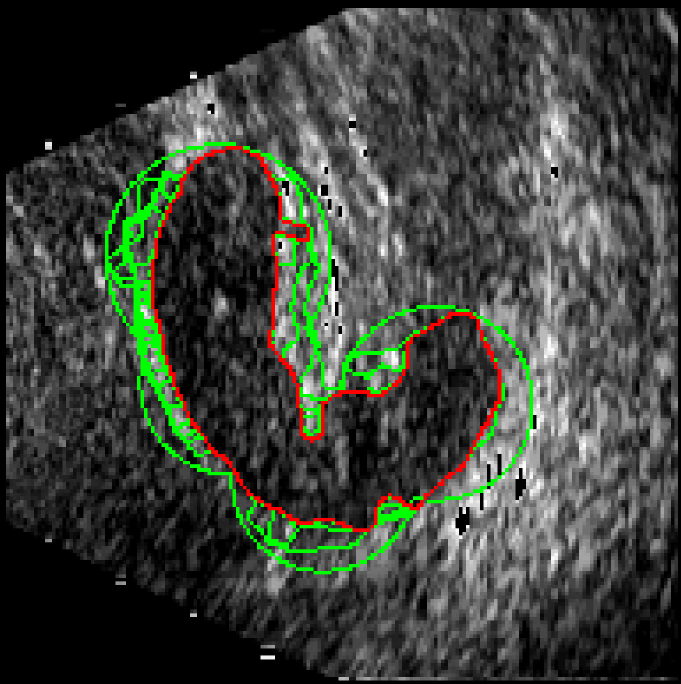
Edge Curves

Segmentation

Segmentation

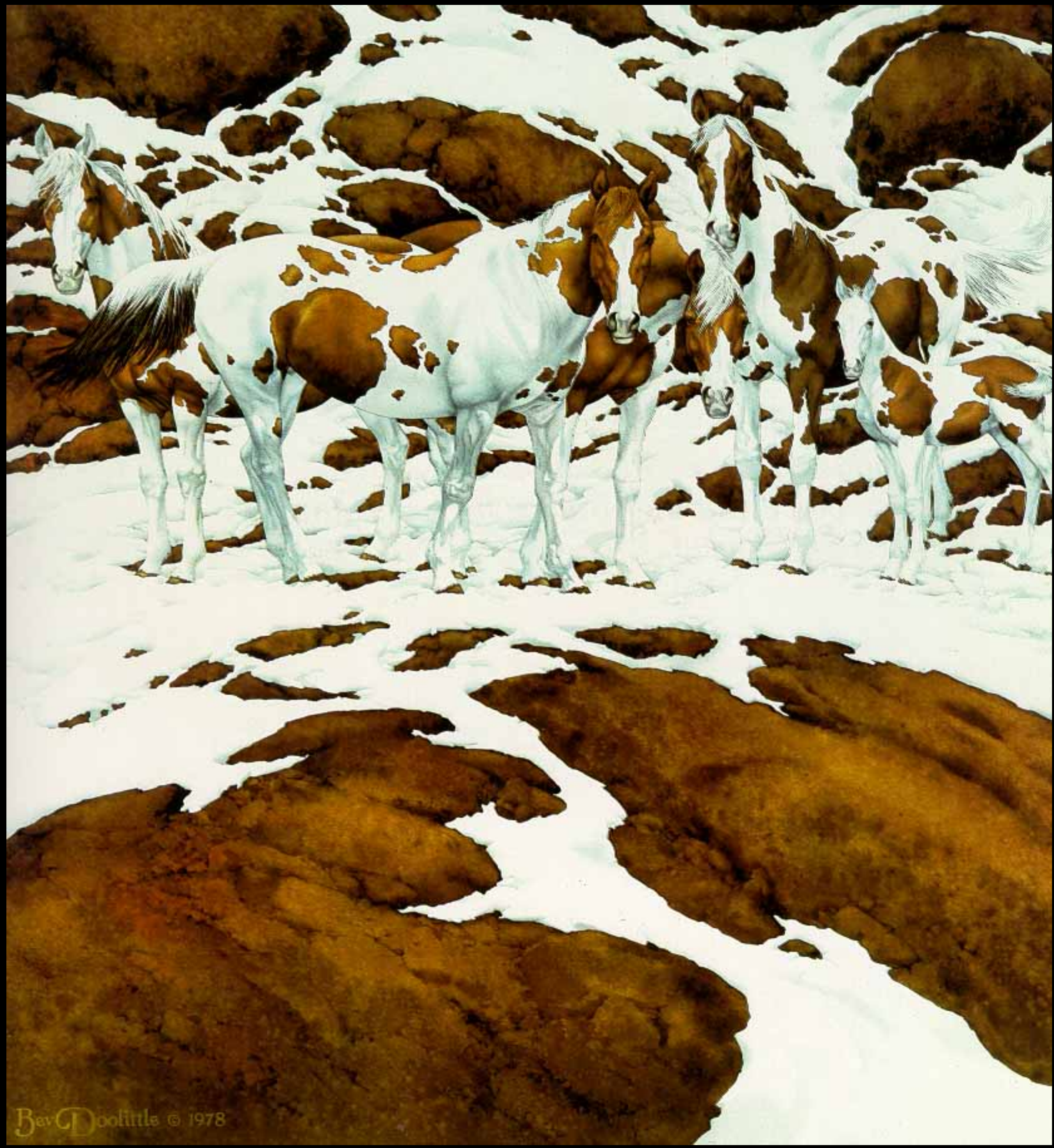
- Ultrasound images





Caselles Kimmel Sapiro *1995*

Pintos



Segmentation

- With a good prior who needs the data…

Wrong Prior???

$$\int g(C)ds \quad \Rightarrow$$

$$\frac{dC}{dt} = \left(g(C)\kappa - \left\langle \nabla g(C), \vec{N} \right\rangle \right) \vec{N}$$

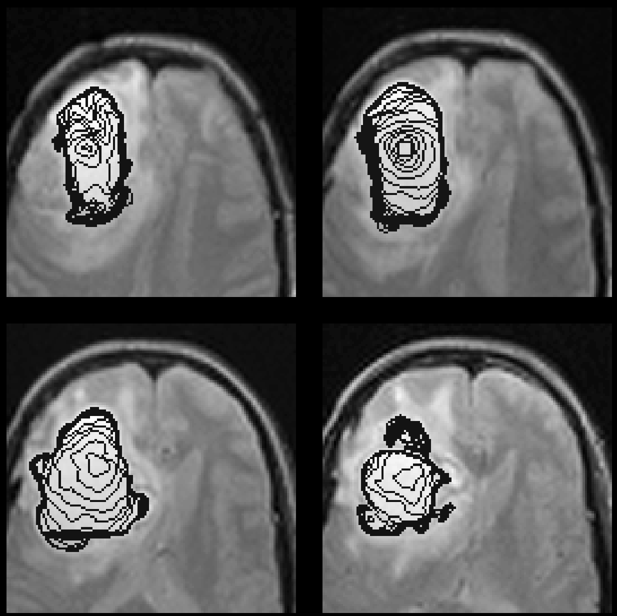
Experiments - Color Segmentation

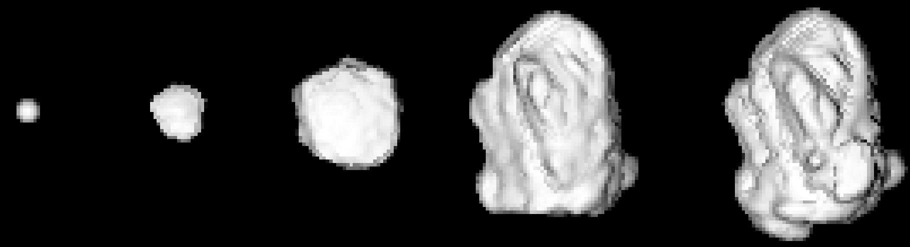
Goldenberg, Kimmel, Rivlin, Rudzsky,
IEEE T-IP 2001

$$\iint g(S)da \Rightarrow$$

$$\frac{dS}{dt} = \left(g(S)H - \left\langle \nabla g(S), \vec{N} \right\rangle\right)\vec{N}$$

Tumor in 3D MRI



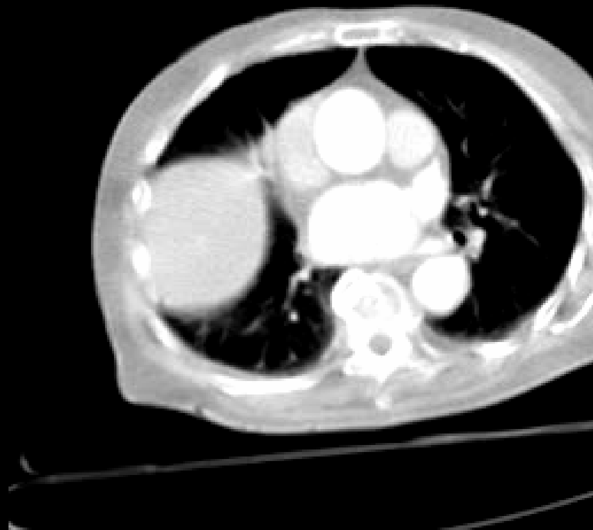


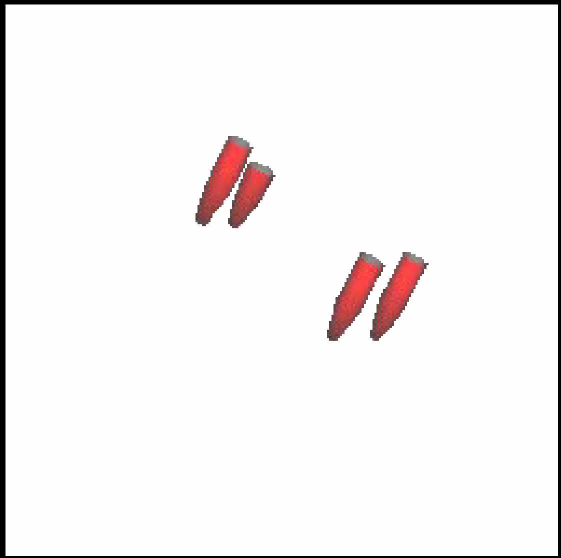
Caselles,Kimmel, Sapiro, Sbert, IEEE T-PAMI 97

$$\iiint g(M)\,dv \quad \Rightarrow$$

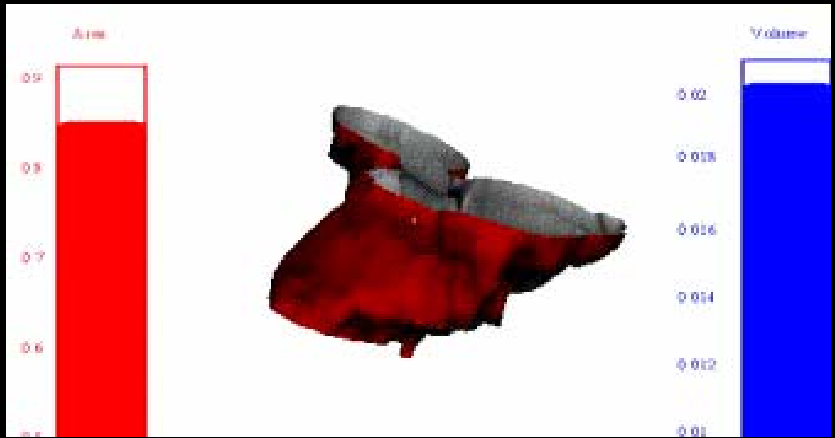
$$\frac{dM}{dt} = \left(g(M)H - \left\langle \nabla g(M), \vec{N} \right\rangle\right)\vec{N}$$

Segmentation in 4D





Malladi, Kimmel, Adalsteinsson,
Caselles, Sapiro, Sethian
SIAM Biomedical workshop 96



$$\int g(C)ds \quad \Rightarrow$$

$$\frac{dC}{dt} = \left(g(C)\kappa - \left\langle \nabla g(C), \vec{N} \right\rangle\right)\vec{N}$$

Tracking in Color Movies

Goldenberg, Kimmel, Rivlin, Rudzsky,
IEEE T-IP 2001

$$\int g(C)ds \quad \Rightarrow$$

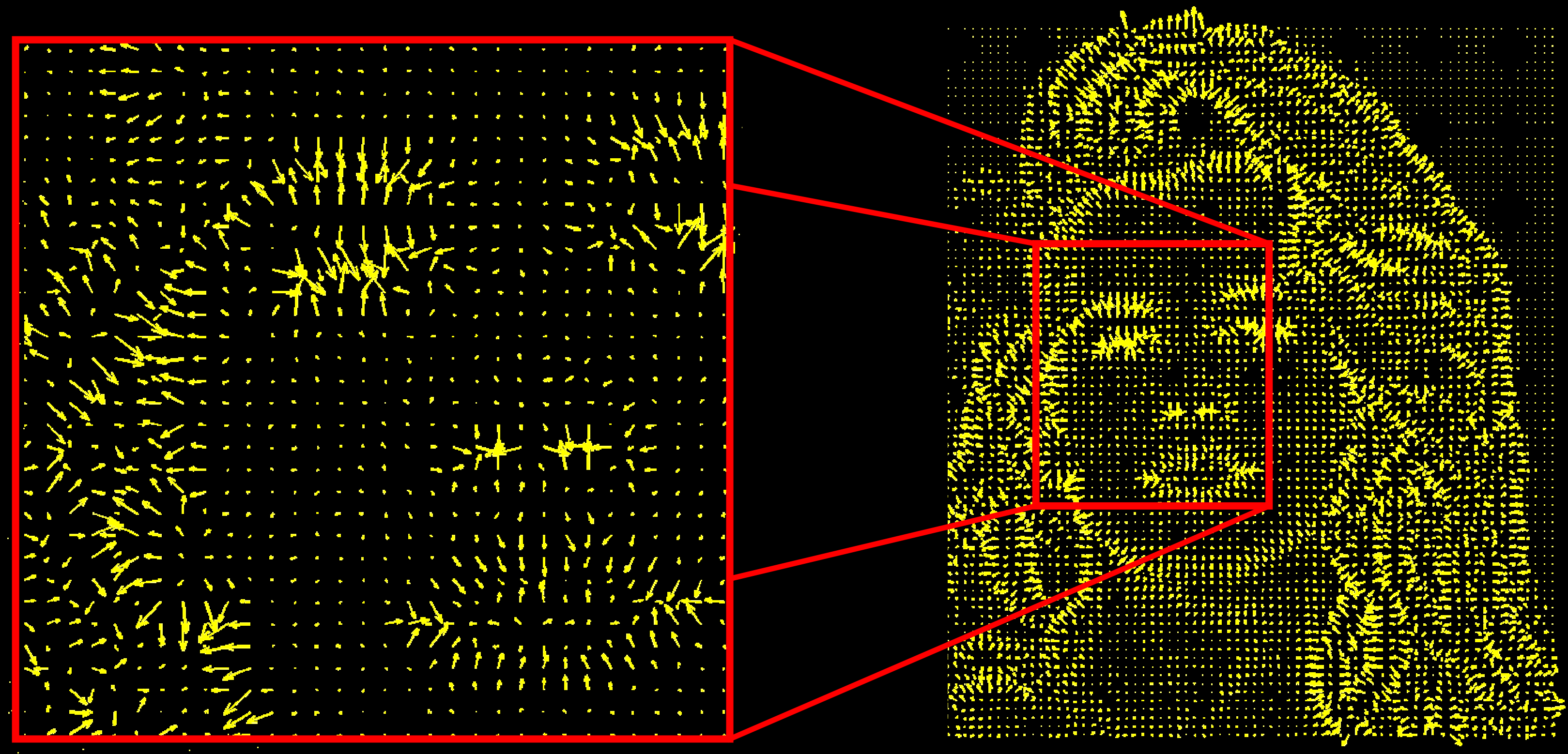
$$\frac{dC}{dt} = \left(g(C)\kappa - \left\langle \nabla g(C), \vec{N} \right\rangle\right)\vec{N}$$

Tracking in Color Movies

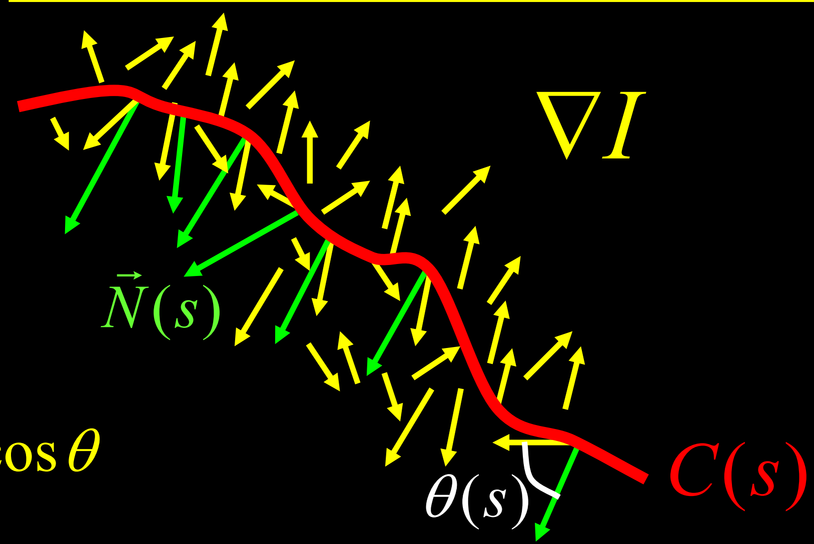
Goldenberg, Kimmel, Rivlin, Rudzsky,
IEEE T-IP 2001

Edge Gradient Estimators

$$I(x, y) \longrightarrow \nabla I$$



Edge Gradient Estimators



$$\langle \vec{N}, \nabla I \rangle = |\nabla I| \cos\theta$$

- We want a curve with large $|\nabla I|$ points and small θ's so:
- Consider the functional $E(C) = \int_C \langle \vec{N}, \nabla I \rangle ds$

The Classic Connection

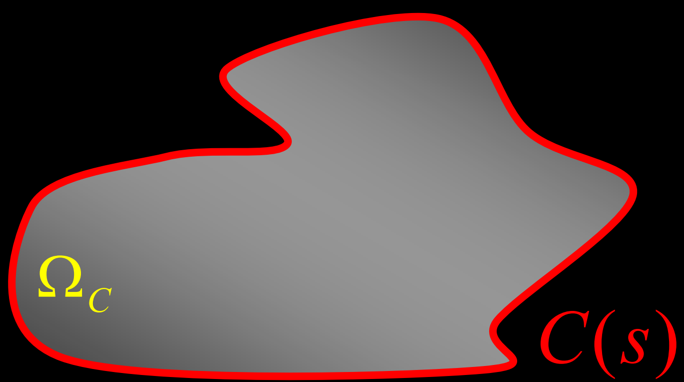
Suppose $\rho(\alpha) = \alpha$ and we consider a closed contour for C(s).
We have

$$E(C) = \oint_{C(s)} \langle \nabla I, \vec{N} \rangle ds$$

and by Green's Theorem we have

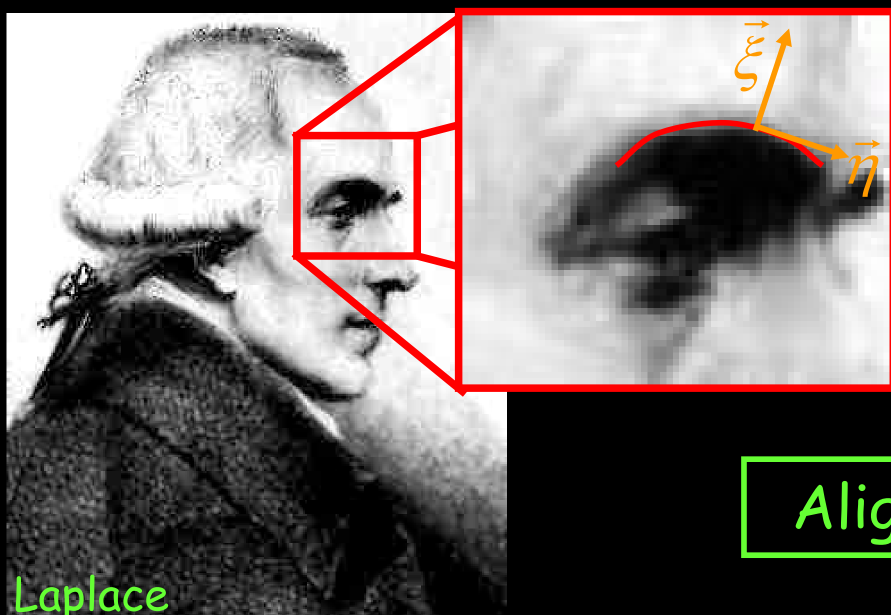
$$= \iint_{\text{Area within } C(s)} \operatorname{div}(\nabla I) dxdy$$

$$= \iint_{\text{Area within } C(s)} \Delta I(x, y) dxdy$$



Haralick/Canny-like Edge Detector

- Haralick suggested $I_{\xi\xi} = 0$ as edge detector



$$\Delta I = I_{xx} + I_{yy} = I_{\xi\xi} + I_{\eta\eta}$$

$$I_{\xi\xi} = \Delta I - I_{\eta\eta}$$

Alignment

Topological Homogeneity

Haralick/Canny Edge Detector $I_{\xi\xi} = 0$

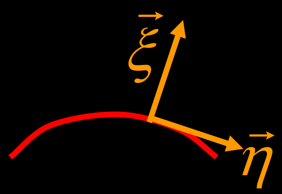
- Haralick $I_{\xi\xi} = \Delta I - I_{\eta\eta}$

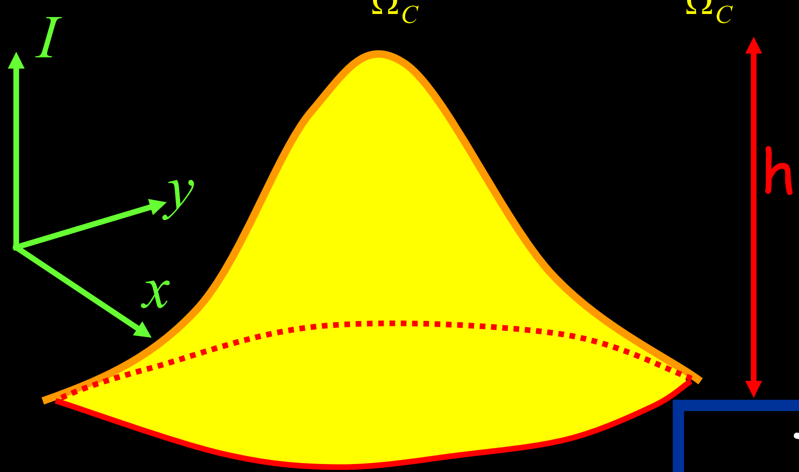
$$I_{\eta\eta} = \mathrm{div}\left(\frac{\nabla I}{|\nabla I|}\right)|\nabla I| = \kappa_I |\nabla I|$$

co-area

Kronrod

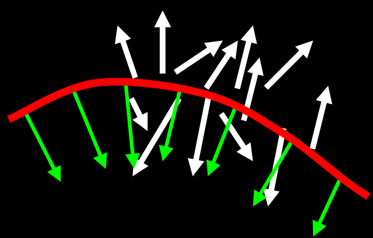
$$\iint_{\Omega_C} I_{\eta\eta}dxdy = \iint_{\Omega_C} \kappa_I |\nabla I| dxdy = \iint_{\Omega_C} \kappa_I dsdI = 2\pi \int dI$$





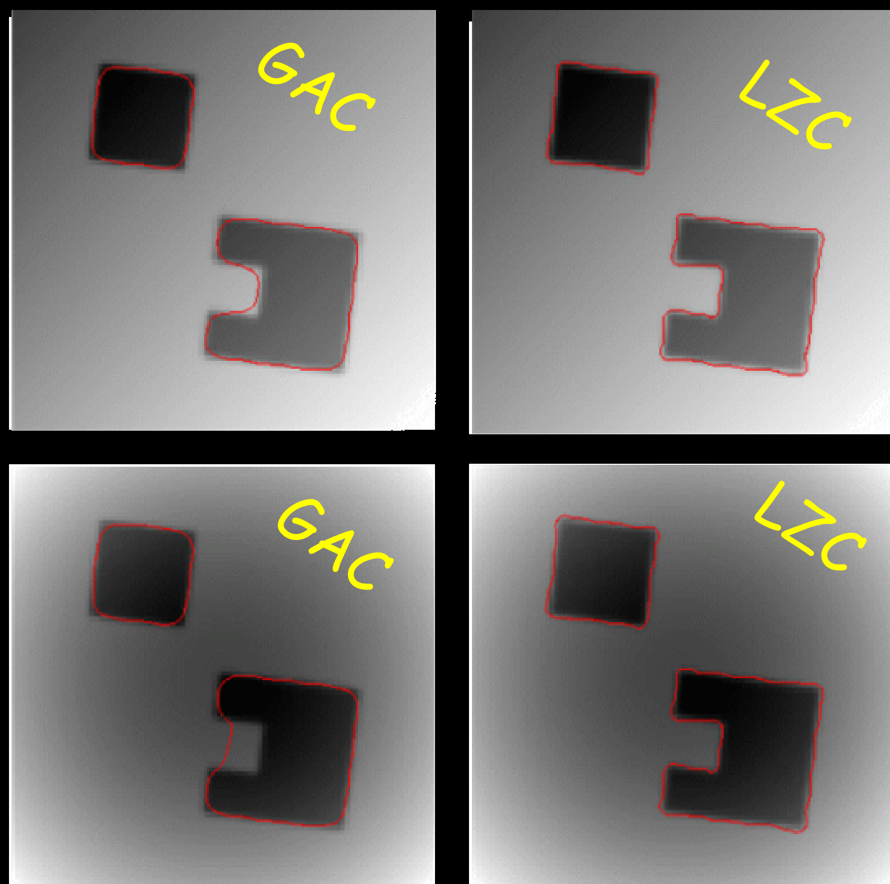
$$\iint_{\Omega_C} I_{\eta\eta}dxdy = 2\pi h$$

Thus, $I_{\xi\xi} = 0$ indicates optimal alignment + topological homogeneity

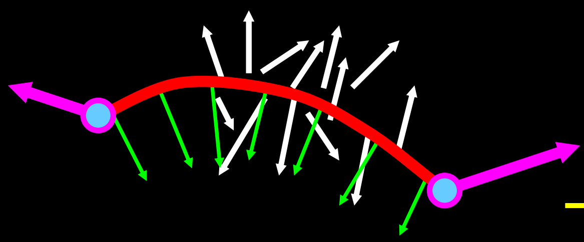


Closed contours $L_\rho + \varepsilon L_g$

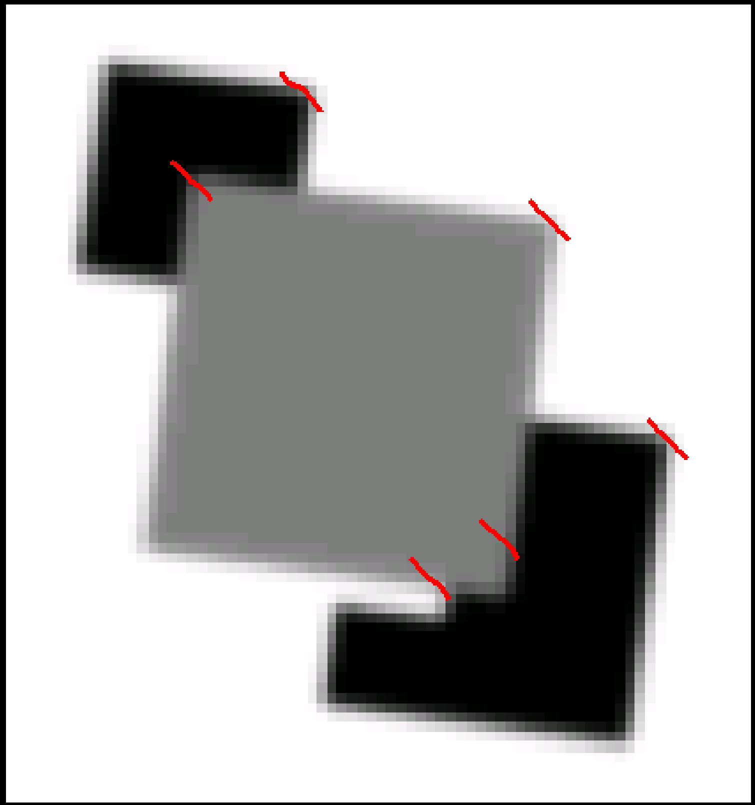
EL eq. $\left(\mathrm{sign}(\langle \vec{N}, \vec{V} \rangle)\mathrm{div}(\vec{V}) + \varepsilon\left(\kappa g - \langle \vec{N}, \nabla g \rangle\right)\right)\vec{N} = 0$



Kimmel-Bruckstein IJCV 2003



Open contours $L_\rho - L$



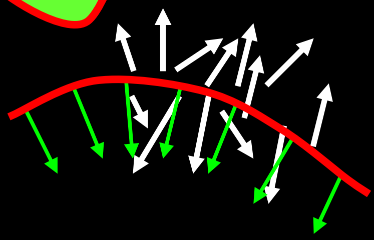
Kimmel-Bruckstein IJCV 2003

Geometric Measures

Weighted arc-length $\int_C g(C(s))ds \Rightarrow (\kappa g - \langle \nabla g, \vec{N} \rangle)\vec{N} = 0$

Weighted area $\iint_\Omega g(x,y)da \Rightarrow g\vec{N} = 0$

Alignment $\int_C \langle \vec{N}, \vec{V} \rangle ds \Rightarrow \mathrm{div}(\vec{V})\vec{N} = 0$



Robust-alignment $\int_C |\langle \vec{N}, \vec{V} \rangle| ds \Rightarrow \mathrm{sign}(\langle \vec{N}, \vec{V} \rangle)\mathrm{div}(\vec{V})\vec{N} = 0$

e.g. $\int_C \langle \vec{N}, \nabla I \rangle ds \Rightarrow \Delta I \vec{N} = 0$

Variational meaning for Marr-Hildreth edge detector

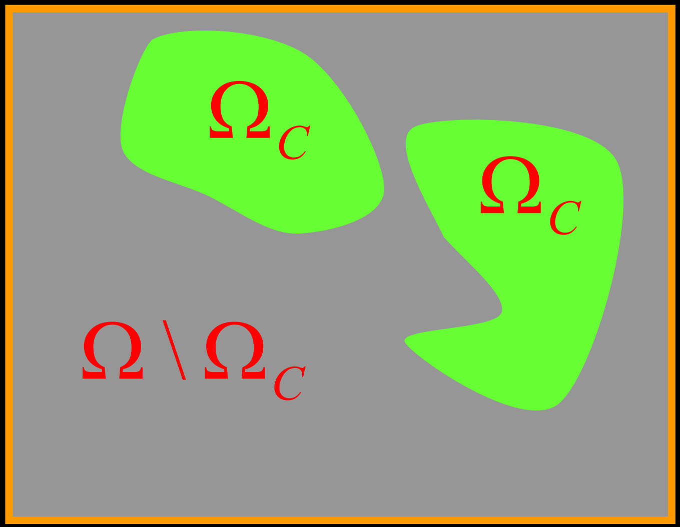
Kimmel-Bruckstein IJCV 2003

Geometric Measures

Minimal variance $$\iint_{\Omega_C} (I - c_1)^2 \, da + \iint_{\Omega \setminus \Omega_C} (I - c_2)^2 \, da$$

Simplified Mumford-Shah, Zhu-Yuille,
Chan-Vese, Max-Lloyd, regularized VQ, Threshold,… $\Longrightarrow$ $(c_1 - c_2)(I - \frac{c_1 + c_2}{2})\vec{N} = 0$

$$\begin{cases} c_1 = \dfrac{\int_{\Omega_C} I da}{\int_{\Omega_C} da} \\ c_2 = \dfrac{\int_{\Omega \setminus \Omega_C} I da}{\int_{\Omega \setminus \Omega_C} da} \\ C_t = (c_2 - c_1)(I - \frac{c_1 + c_2}{2})\vec{N} = 0 \end{cases}$$

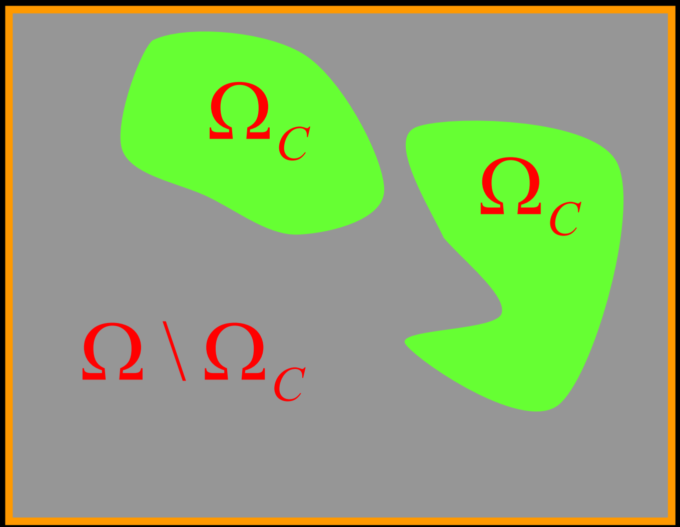


Geometric Measures

Robust minimal deviation $\iint_{\Omega_C} |I - c_1| da + \iint_{\Omega \setminus \Omega_C} |I - c_2| da$

$$\Rightarrow (|I - c_1| - |I - c_2|)\vec{N} = 0$$

$$\begin{cases} c_1 = \underset{\Omega_C}{\text{median}}(I(x,y)) \\ c_2 = \underset{\Omega \setminus \Omega_C}{\text{median}}(I(x,y)) \\ C_t = (|I - c_2| - |I - c_1|)\vec{N} \end{cases}$$



Tracking

Goldenberg, Kimmel, Rivlin, Rudzsky,
ECCV 2002

Tracking

Goldenberg, Kimmel, Rivlin, Rudzsky,
ECCV 2002

Tracking

Goldenberg, Kimmel, Rivlin, Rudzsky,
ECCV 2002

Information extraction

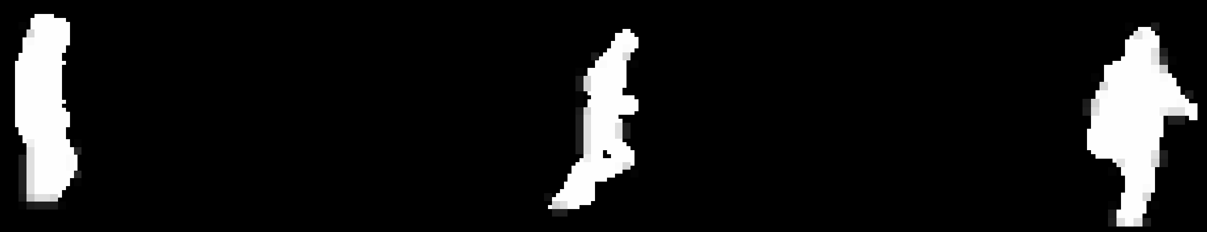
Goldenberg, Kimmel, Rivlin, Rudzsky,
ECCV 2002

Classification (dogs & cats)

walk run gallop cat...

Goldenberg, Kimmel, Rivlin, Rudzsky,
ECCV 2002

Classification (people)

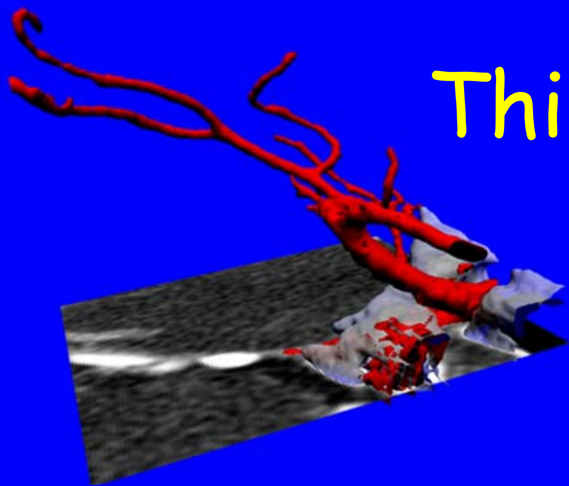


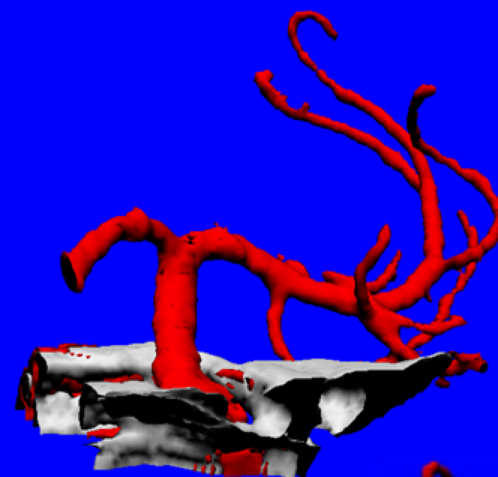
walk run run45

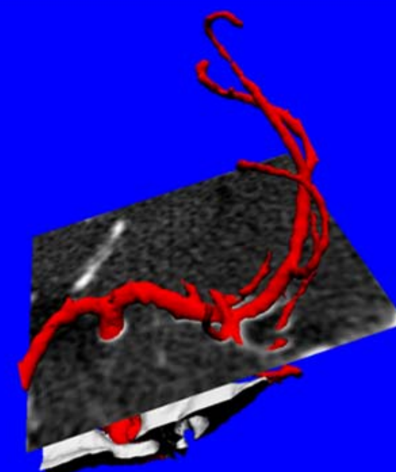
Goldenberg, Kimmel, Rivlin, Rudzsky,
ECCV 2002

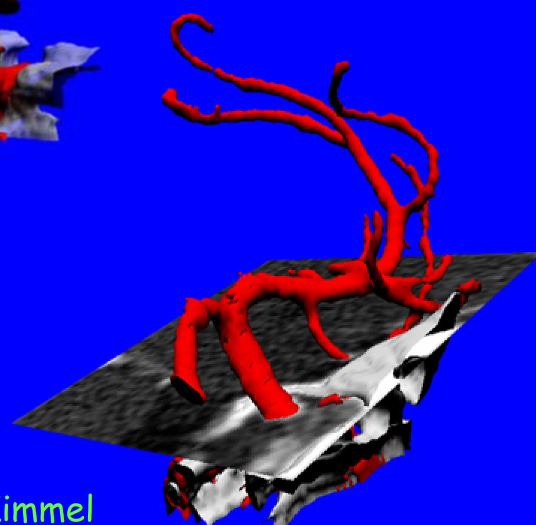
Thin Structures

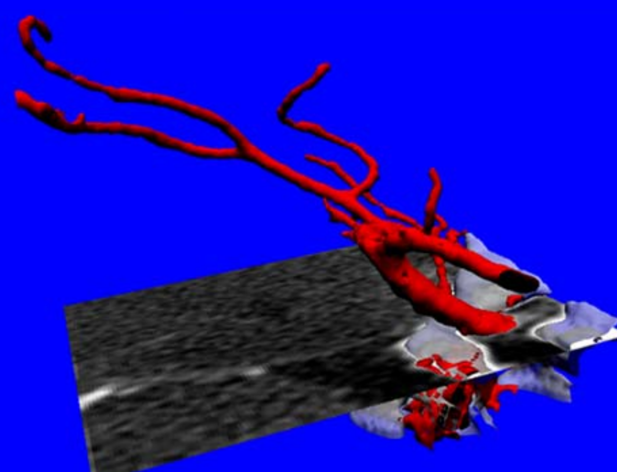
$$I_{\xi\xi} = \Delta I - I_{\eta\eta} - I_{\gamma\gamma}$$
$$= \Delta I - H\left|\nabla I\right|$$





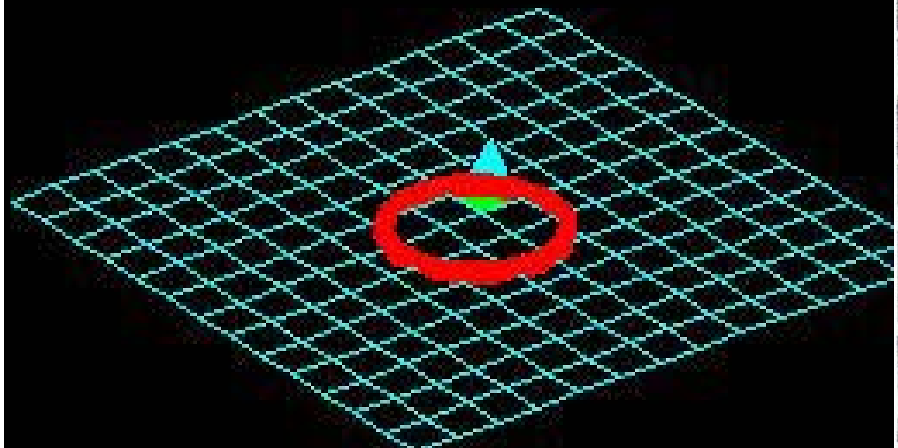




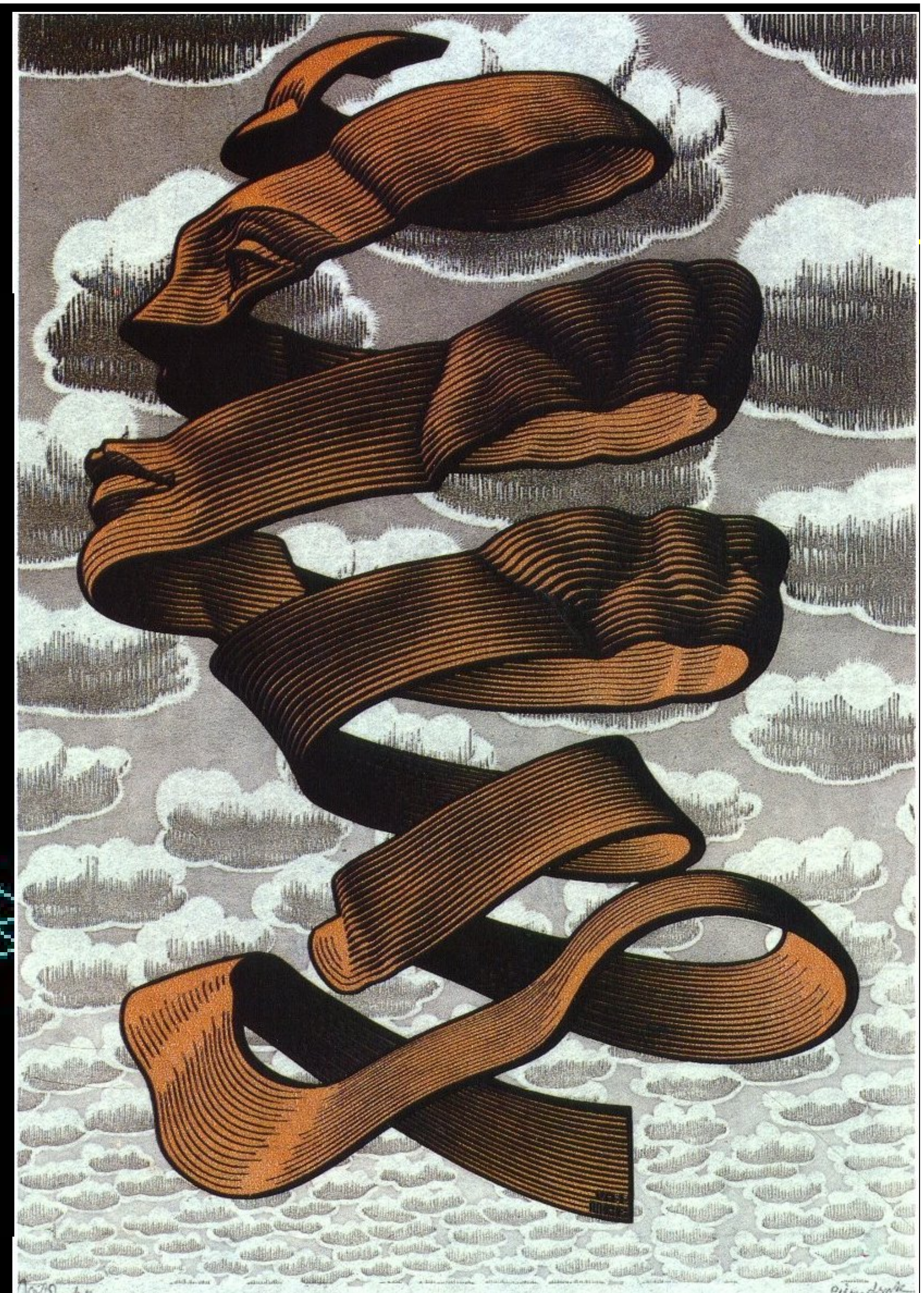


Holzman-Gazit, Goldshier, Kimmel
MICCAI 2003

Distance maps and minimal geodesics



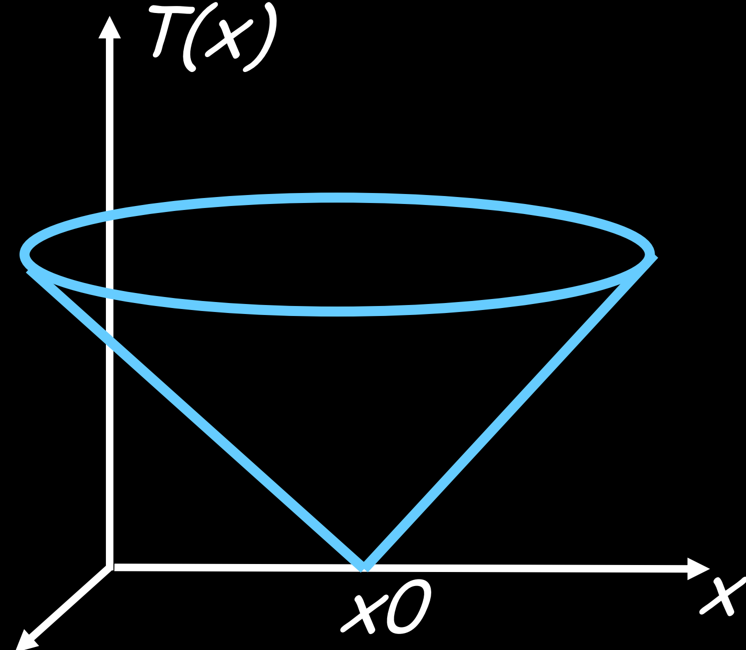
www.math.berkeley.edu/~sethian



Escher, Rind 1955

Measuring geodesic distances: Fast Marching on Triangulated Domains

- Find distance *T(x)*, given *T(x0)=0*.



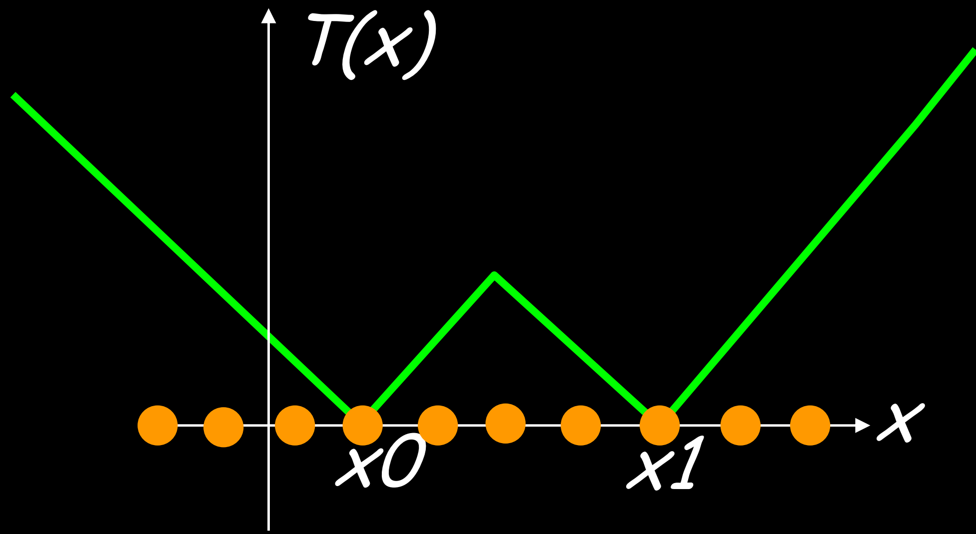
- Solution: *T(x)=|x-x0|*.
- $\left|\frac{d}{dx}T(x)\right| = 1$ except *x0*.
- Or in 2D $\left|\nabla T(x,y)\right| = 1$ where

$$\left|\nabla T(x,y)\right|^2 \equiv \left(\tfrac{d}{dx}T\right)^2 + \left(\tfrac{d}{dy}T\right)^2$$

Consistent Approximation

$$\left|\frac{d}{dx}T(x)\right| = 1$$

- $\left|\frac{d}{dx}T(x)\right| \approx \left|\max\{(T_i - T_{i-1})/h, (T_i - T_{i+1})/h, 0\}\right|$
- Updated *i* has always $T_i > \min\{T_{i-1}, T_{i+1}\}$
- `upwind' from where the `wind blows'

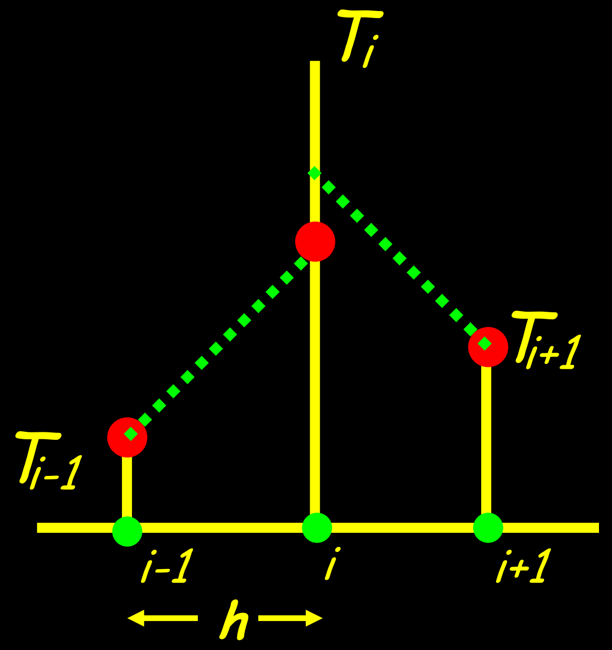


Update Procedure

Set $T_i = \infty$, and $T(x0)=T(x1)=0$.

REPEAT UNTIL convergence,

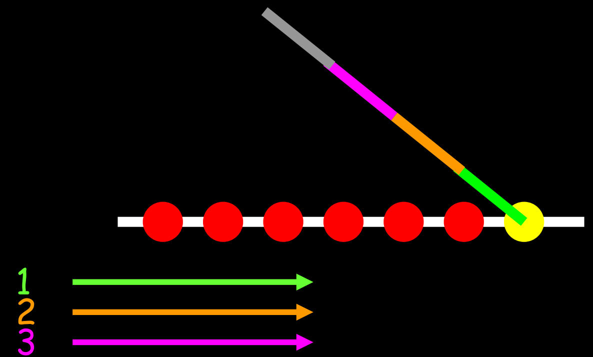
- FOR each i
  - $m = \min\{T_{i-1}, T_{i+1}\}$
  - $T_i = \min\{T_i, m + h\}$



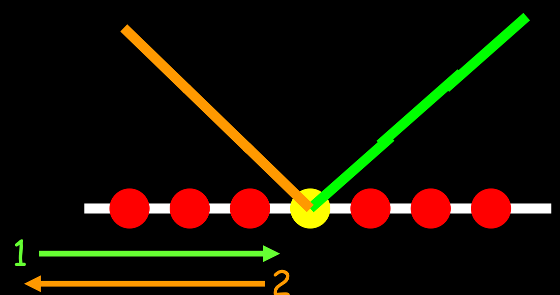
Update Order

What is the optimal order of updates?

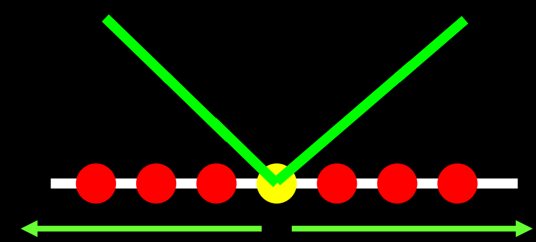
Solution I: Scan the line successively left to right. *N* scans, i.e. $O(N^2)$



Solution II: Left to right followed by right to left. Two scans are sufficient. (Danielson`s distance map 1980)



Solution III: Start from *x0*, update its neighboring points, accept updated values, and update their neighbors, etc.



2D Approximation

Initialization:

- $\forall \{i, j\} : T_{ij} =$ given initial value or ∞

Update:

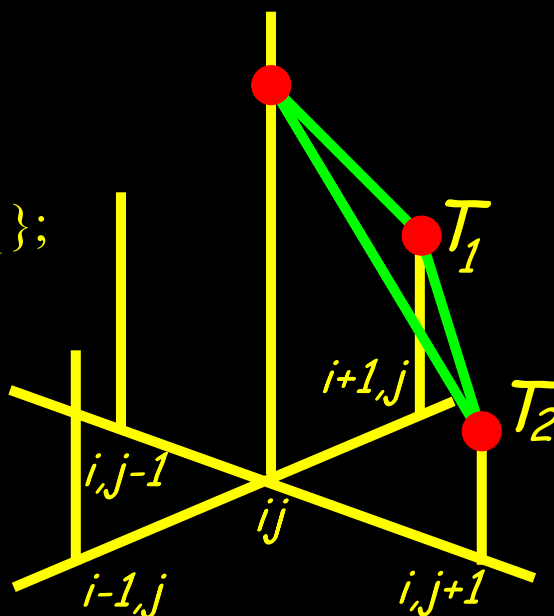
- $T_1 = \min\{T_{i-1,j}, T_{i+1,j}\}; \quad T_2 = \min\{T_{i,j-1}, T_{i,j+1}\};$

IF $(|T_1 - T_2| < 1)$ THEN

$$T = \frac{T_1 + T_2 + \sqrt{2 - (T_1 - T_2)^2}}{2};$$

ELSE $T = \min\{T_1, T_2\} + 1;$

$T_{ij} = \min\{T_{ij}, T\};$



Fitting a tilted plane with gradient 1, and two values anchored at the relevant neighboring grid points.

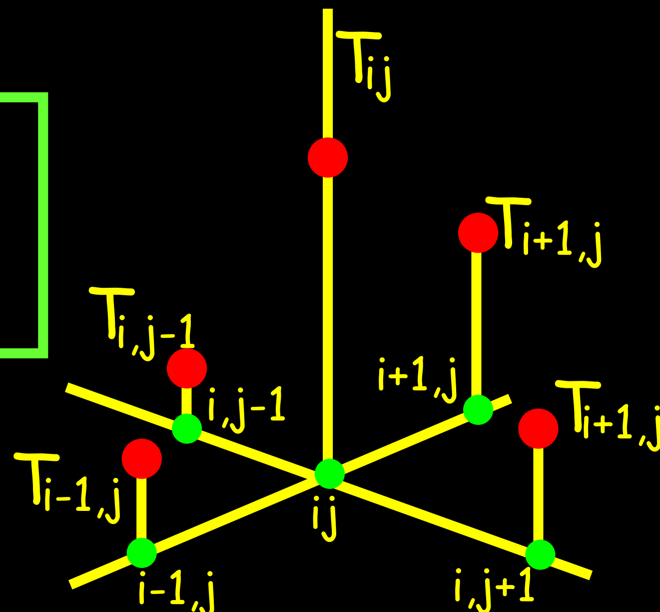
Upwind approximation in 2D

$$\left(\max\{ D^{-x}{}_{ij}T, -D^{+x}{}_{ij}T, 0\}\right)^2 + \left(\max\{ D^{-y}{}_{ij}T, -D^{+y}{}_{ij}T, 0\}\right)^2 = 1,$$

where $$D^{-x}{}_{ij}T = \frac{(T_{ij} - T_{i-1,j})}{h}$$

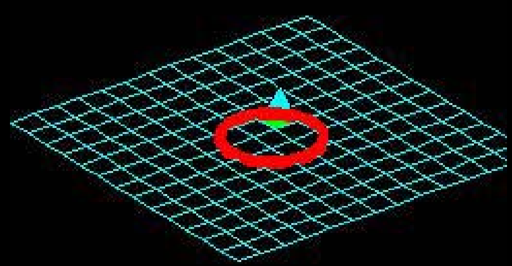
$\Leftrightarrow$

$$\left(\max\{T_{ij} - \min\{T_{i-1,j}, T_{i+1,j}\}, 0\}\right)^2 + \left(\max\{T_{ij} - \min\{T_{i,j-1}, T_{i,j+1}\}, 0\}\right)^2 = h^2,$$



Computational Complexity

- *T* is systematically constructed from smaller to larger *T* values.
- Update of a heap element is *O(log N)*.
- Thus, upper bound of the total is *O(N log N)*.



www.math.berkeley.edu/~sethian

Edge Integration

Cohen-Kimmel, IJCV, 1997.

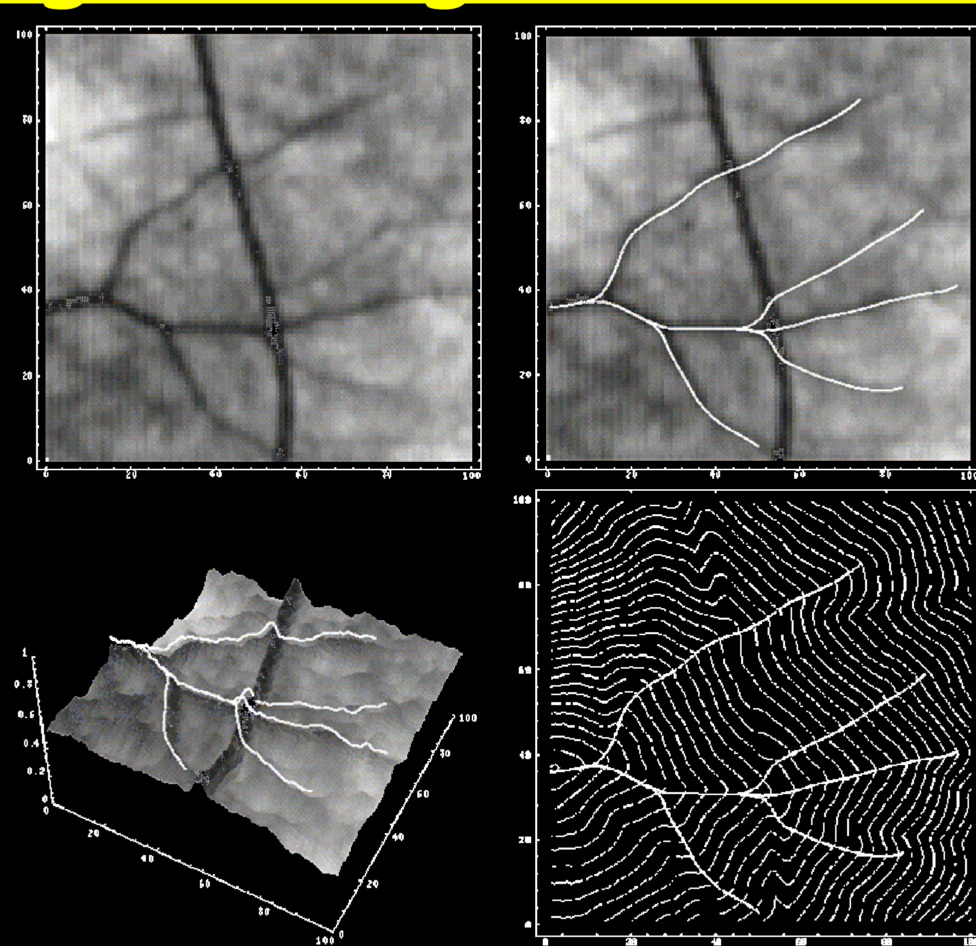
Solve the 2D Eikonal equation

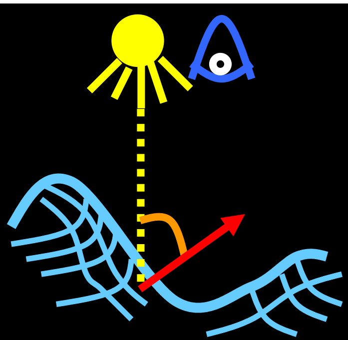
$$\left(D_x T\right)^2 + \left(D_y T\right)^2 = F_{ij}^2$$

given $T(p)=0$

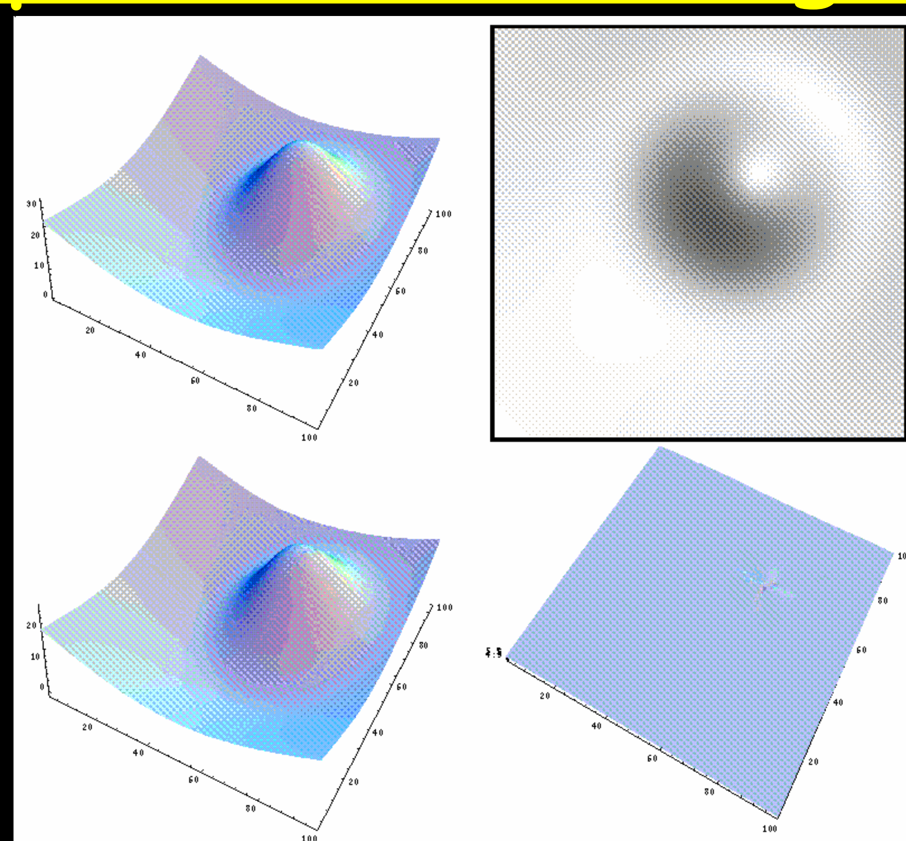
Minimal geodesic w.r.t.

$$ds^2 = F^2(I)\left(dx^2 + dy^2\right)$$





Shape from Shading



Rouy-Tourin SIAM-NU 1992,
Kimmel-Bruckstein CVIU 1994,
Kimmel-Sethian JMIV 2001.

Solve the 2D Eikonal equation

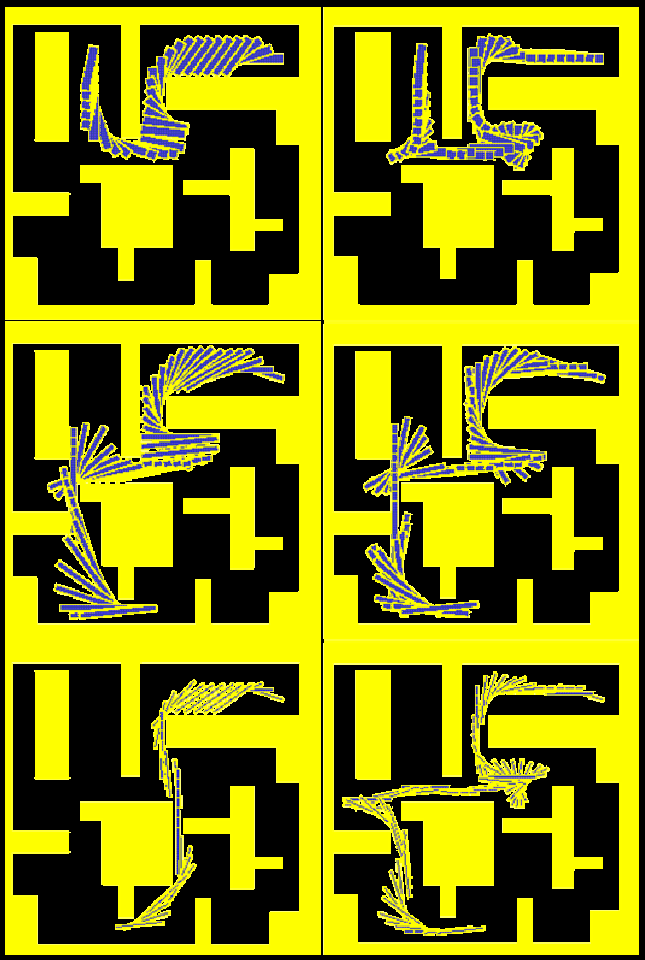
$$(D_x T)^2 + (D_y T)^2 = F_{ij}^2$$

where $F^2 = (1 - I^2)/I^2$

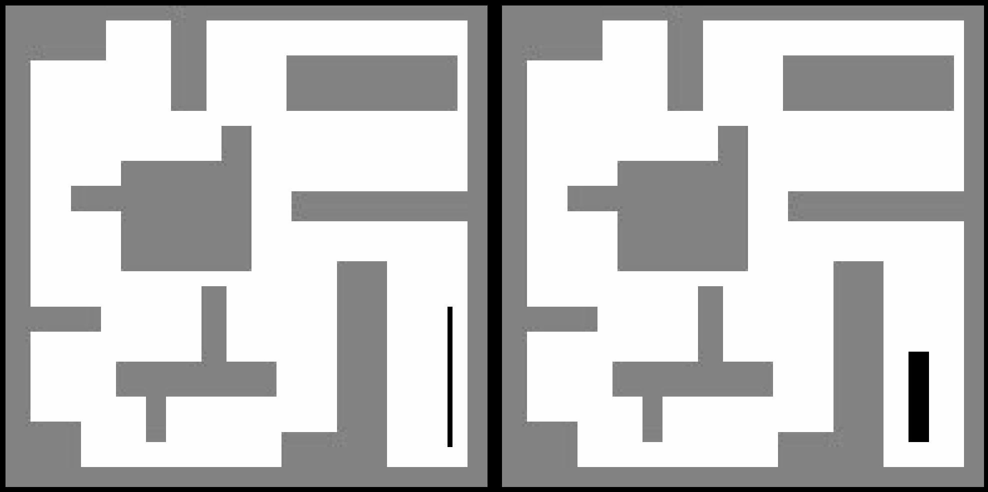
Minimal geodesic w.r.t.

$$ds^2 = F^2(I)\left(dx^2 + dy^2\right)$$

Path Planning 3 DOF



$$(D_x T)^2 + (D_y T)^2 + (D_\varphi T)^2 = F_{ijk}^2$$



$$ds^2 = F^2(x, y, \varphi)\left(dx^2 + dy^2 + d\varphi^2\right)$$

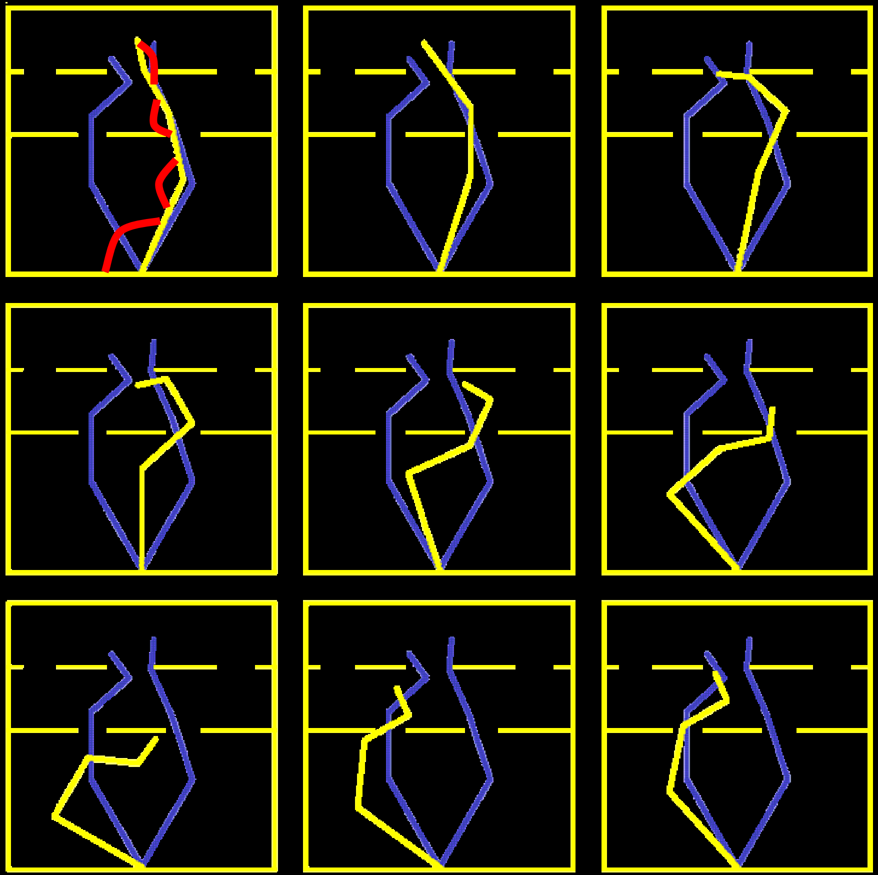
Path Planning 4 DOF

Solve the Eikonal Eq. in 4D

$$\left(D_{\varphi_1}T\right)^2+\left(D_{\varphi_2}T\right)^2+\left(D_{\varphi_3}T\right)^2+\left(D_{\varphi_4}T\right)^2=F_{ijkl}^2$$

Minimal geodesic w.r.t.

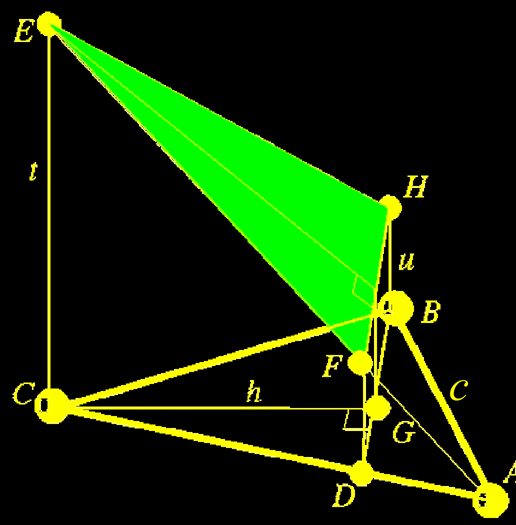
$$ds^2=F^2(\varphi_1,\varphi_2,\varphi_3,\varphi_4)\left(d\varphi_1^2+d\varphi_2^2+d\varphi_3^2+d\varphi_4^2\right)$$

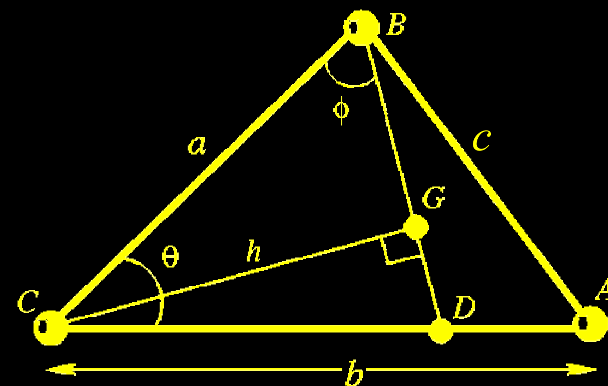


Update Acute Angle

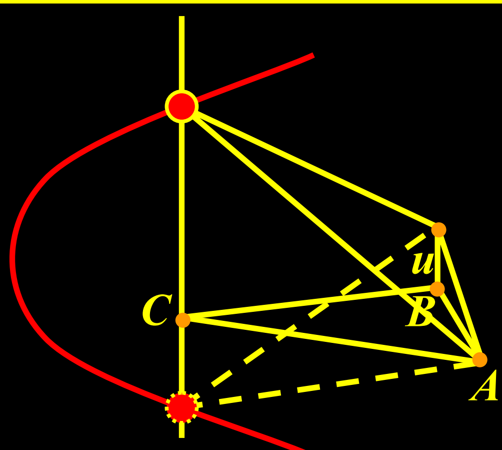
Given *ABC*, update *C*.
Consistency and monotonicity:
Update only `from within the triangle' *h* in *ABC*
Find *t=EC* that satisfies the gradient approximation

(t-u)/h= 1.





Update Procedure



We end up with:

$$c^2 t^2 + 2bu(a^2\cos\theta - b)t + b^2\left(u^2 - (a\sin\theta)^2\right) = 0$$

$$\alpha t^2 + \beta t + \chi = 0$$

t must satisfy *u<t*, and *h* in *ABC*.

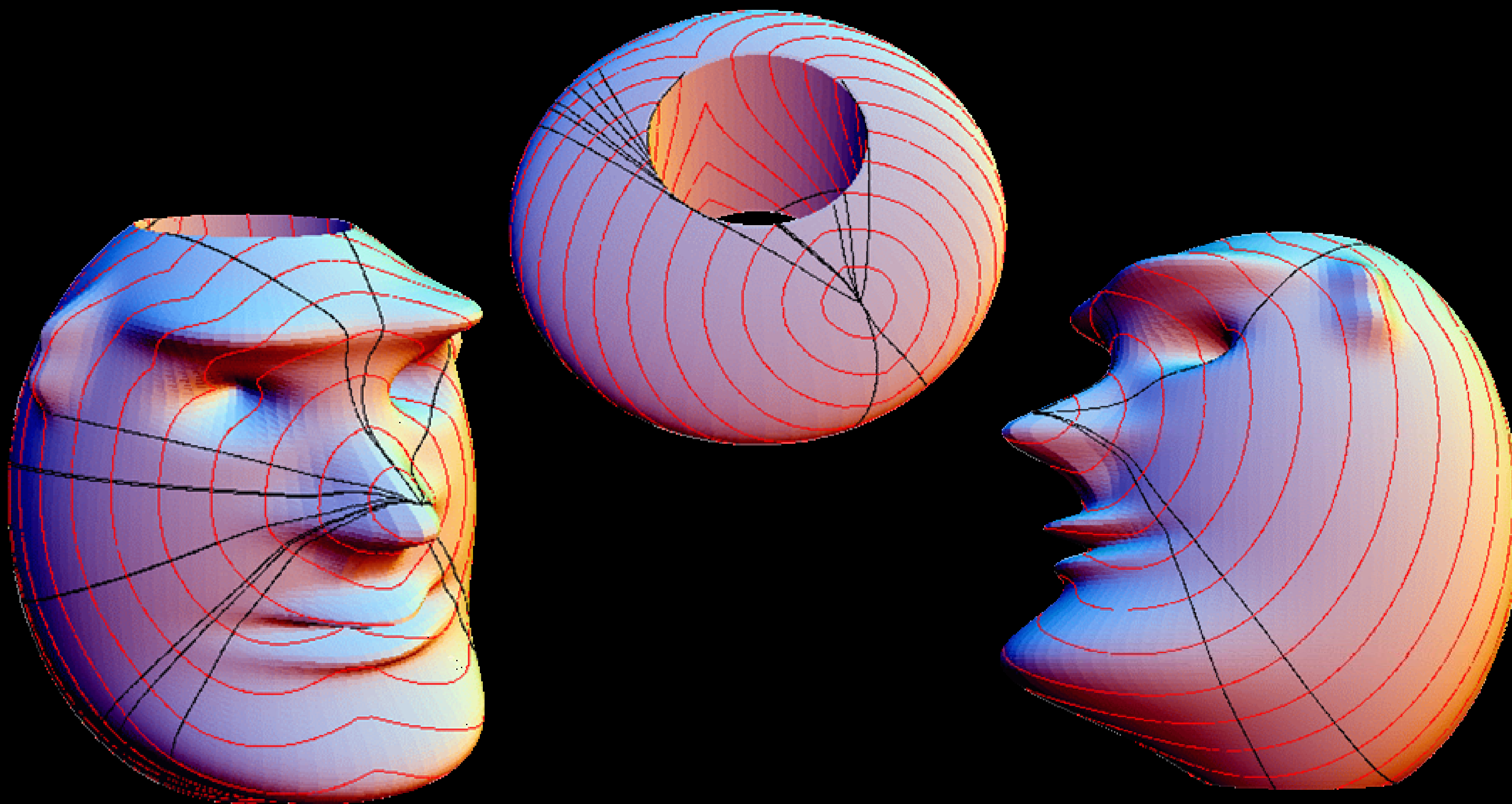
The update procedure is

IF *(u<t)* AND *(a cos θ < b(t-u)/t < a/cosθ)*

THEN *T(C) = min {T(C),t+T(A)};*

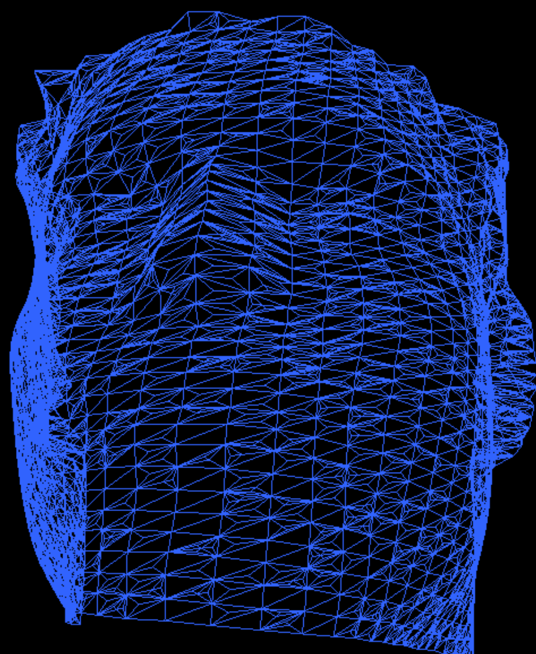
ELSE *T(C)= min {T(C),b+T(A),a+T(B)}.*

Minimal Geodesics

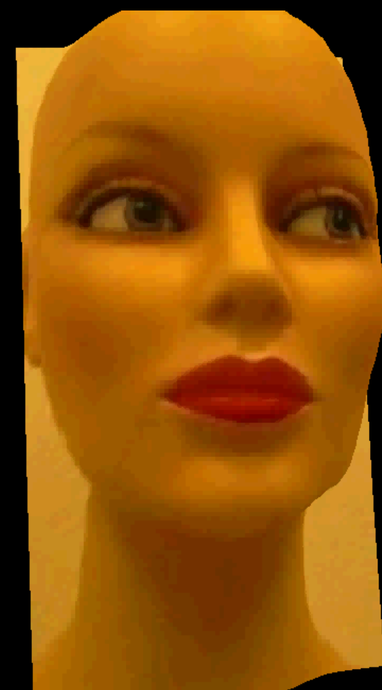


Kimmel and Sethian, PONAS 1998

More Applications



re-triangulation



semi-manual segmentation

halftoning in 3D

Adi, Kimmel 2002